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Budgeted Prize-Collecting Traveling Salesman and Minimum Spanning Tree Problems

Alice Paul
Data Science Initiative, Brown University, Providence, RI, USA, alice_paul@brown.edu

Daniel Freund
Center for Applied Mathematics, Cornell University, Ithaca, NY, USA, df365@cornell.edu

Aaron Ferber
Operations Research and Information Engineering, Cornell University, Ithaca, NY, USA, amf272@cornell.edu

David B. Shmoys, David P. Williamson
Operations Research and Information Engineering, Cornell University, Ithaca, NY, USA,
{dbs10@cornell.edu, davidpwilliamson@cornell.edu}

We consider constrained versions of the prize-collecting traveling salesman and the prize-collecting minimum spanning tree problems. The goal is to maximize the number of vertices in the returned tour/tree subject to a bound on the tour/tree cost. Rooted variants of the problems have the additional constraint that a given vertex, the root, must be contained in the tour/tree. We present a 2-approximation algorithm for the rooted and unrooted version of both the tree and the tour variant. The algorithm is based on a parameterized primal-dual approach. It relies on first finding a threshold value for the dual variable corresponding to the budget constraint in the primal and then carefully constructing a tour/tree that is, in a precise sense, just within budget. We improve upon the best-known guarantee of $2+\epsilon$ for the rooted and unrooted tour version and $3+\epsilon$ for the rooted tree version; for the unrooted case (to the best of our knowledge) no approximation algorithm was previously known. Our analysis extends to the setting with weighted vertices, in which we want to maximize the total weight of vertices in the tour/tree. Interestingly enough, the algorithm and analysis for the rooted and the unrooted case are almost identical.

Key words: approximation algorithms, traveling salesman problem, constrained optimization
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1. Introduction In the classical traveling salesman problem, we are given an undirected graph $G=(V,E)$ with edge costs $c_e \geq 0$ for all $e \in E$. The goal is to construct a tour visiting all vertices in the graph while minimizing the cost of edges in the tour. If, however, we are given a bound on the cost of the tour, then we may not be able to visit all vertices. In particular, suppose that we are given a budget $D \geq 0$. In the **budgeted prize-collecting traveling salesman problem**, a feasible tour is a multiset of edges F such that (a) F is a tour on a subset $S \subseteq V$ and (b) the cost of the edges in F is at most D. The goal is to find a feasible tour F that maximizes $|S|$, the number of vertices visited. Here, we do not require the graph to be complete and allow a tour to visit nodes more than once. Similarly, in the **budgeted prize-collecting minimum spanning tree problem**, a feasible tree is a set of edges T such that (a) T specifies a tree spanning a subset

$S \subseteq V$ and (b) the cost of the edges in T is at most D. Again, the goal is to find a feasible tree T that maximizes $|S|$. For either problem, we may consider the rooted versions in which we are also given a vertex r with the added constraint that it must be contained in the returned solution. We say the problem is unrooted if there is no root r given.

The budgeted version of the traveling salesman problem arises naturally in many routing problems that have a distance or time constraint. For example, a bikeshare system may have bike stations located around a city that need repair. Throughout the day, the system operator wants to route a repairman over his work period while maximizing the number of stations that receive maintenance (in fact, this precise question emerged from our ongoing work with New York City Bikeshare [12]). We can represent this problem as a budgeted prize-collecting traveling salesman problem. Further, we can also capture the setting in which stations have varying importance; we discuss in Section 8 how to extend our algorithm to a setting in which vertices have weights and the goal is to maximize the weight of vertices visited. In Section 9, we apply our algorithm to such instances using Citi Bike data in New York City. The budgeted version of the minimum spanning tree also arises in a range of applications, including telecommunication network design problems in which an infrastructure budget is weighed against the number of customers served.

In this paper, we present a parameterized primal-dual algorithm that achieves an approximation guarantee of 2 for all the problems mentioned above, including both the rooted and unrooted versions. The algorithm is based on a primal-dual subroutine which uses a linear programming relaxation of the problem. First, we search for a "good" value for the dual variable corresponding to the budget constraint in the primal. Having set this variable, we can then increase the other dual variables and form a forest of edges whose corresponding dual constraint is tight. For the tour problem, we then choose a tree in this forest and carefully prune it so that doubling this tree forms a tour that, in a precise sense, will be just within budget. For the tree problem, we prune edges such that the tree itself is just within budget. Lastly, we show that either our constructed tour/tree is within a factor of 2 of optimal or we recurse on a smaller subgraph.

Literature Review There have been many prize-collecting variants of both the traveling salesman problem (TSP) and the minimum spanning tree problem (MST) that seek to balance the number of vertices in the tree or tour with the cost of edges used. Johnson, Minkoff, and Phillips [17] characterize four main variants of prize-collecting MST problems: the Goemans-Williamson Minimization problem that minimizes the cost of edges plus a penalty for vertices not in the tree (this is the traditional prize-collecting objective), the Net Worth Maximization problem that maximizes the weight of vertices in the tree minus the cost of used edges, the Quota problem that minimizes the cost of a tree containing at least Q vertices, and, finally, the Budget problem that maximizes the number of vertices in the tree subject to the cost of the tree being at most D. All of the variants above can be extended to a corresponding TSP version that constructs a tour rather than a tree.

Our algorithm is most similar to that of Garg [14], who presents a 2-approximation algorithm for the Quota problem for MST, improving upon the previous results of Garg [13], Arya and Ramesh [2], and Blum, Ravi, and Vempala [4]. Johnson et al. [17] observe that a 2-approximation algorithm for the Quota problem yields a $(3+\epsilon)$-approximation algorithm for the corresponding unrooted Budget problem. To our knowledge this was the previously best-known guarantee for the unrooted MST variant. Prior to Garg's result, Levin [19] proved a $(4+\epsilon)$-approximation algorithm. Our 2-approximation algorithm for the budgeted prize-collecting MST thus improves upon the best known approximation ratio for the unrooted version. Though the rooted version is mentioned by Johnson et al. [17], to the best of our knowledge no guarantee for it was known prior to this work. From a technical perspective, the main distinction between our work and that of Garg [14] lies in how we find the threshold value for the dual variable; further, our overall proof relies on more precise accounting.

For the Goemans-Williamson Minimization problem for MST, Archer et al. [1] obtain a $(2-\epsilon)$-approximation guarantee, improving upon the long-standing bound of 2 obtained by Goemans and Williamson [15] in 1995. Further, Archer et al. [1] successfully applied this algorithm to telecommunication network problems. Dilkina and Gomes [9] also study a variant of the problem in the context of wildlife conservation using mixed-integer programming. Lastly, Feigenbaum et al. [10] show the Net Worth Maximization problem for MST is NP-hard to approximate within any constant.

To the best of our knowledge, the previous best approximation guarantee for the rooted and unrooted budgeted prize-collecting TSP arises from a special case of a result by Chekuri, Korula, and Pál [6]. Their work provides a $(2+\epsilon)$-approximation algorithm with running-time $n^{O\left(\frac{1}{\epsilon^2}\right)}$ for the more general orienteering problem, in which the goal is to find an $s-t$ path, where s and t are given, with bounded cost that maximizes the number of vertices visited on the path. By setting $s=t=r$ this then yields a $(2+\epsilon)$-approximation algorithm for the rooted budgeted prize-collecting TSP. For the unrooted case, one can iterate over all possible nodes as roots. In contrast, as we show in Section 8, our approach to the unrooted version does not rely on iterating over all possible roots. The orienteering problem itself has attracted much attention within the combinatorial optimization community, with other variants studied by [22], [5], [8], [7], and [16].

There exist other adaptations of prize-collecting problems not discussed above. Specifically, Ausiello, Demange, Laura, and Paschos [3] present a 2-approximation algorithm for an on-line variant of the Quota problem for the TSP. Frederickson and Wittman [11] study the so-called traveling repairmen problem, in which each vertex can only be visited within a specific time window and the goal is to either maximize the number of vertices visited within a certain time period or to minimize the time visiting all vertices; they give constant-factor approximation algorithms for both variations. Lastly, Nagarajan and Ravi [20] study the problem of minimizing the number of tours to cover all vertices subject to each tour having bounded distance. They give a 2-approximation algorithm for tree metric distances.

The paper is structured as follows. In Section 2, we present the linear programming (LP) relaxation for the rooted version of the budgeted prize-collecting traveling salesman problem. In Section 3, we use this LP to develop a primal-dual subroutine that will inform our decisions. In Section 4, we use this subroutine to present an outline of the entire parameterized primal-dual algorithm and the proof of its approximation ratio, providing some intuition behind what type of tour will be near optimal. In Section 5, we prove an upper bound on the size of an optimal solution. Then, in Section 6, we show how to set the dual variable corresponding to the budget constraint, and in Section 7, we show how to construct our proposed tour that is within a factor of 2 of optimal. For ease of presentation, we present these results for the rooted version budgeted prize-collecting traveling salesman problem with unit weights and show how the analysis extends to the weighted case and the MST version in Section 8. In Section 8, we also show how our algorithm tackles the unrooted versions of the two problems without having to run the rooted version on all possible roots. Last, we present computational experiments in Section 9.

2. LP Formulation As mentioned, we present our result in terms of the *rooted* budgeted prize-collecting traveling salesman problem. In this version of the problem, there is the added constraint that the returned tour must contain a root node $r \in V$. However, in Section 8, we show that our algorithm (and analysis) directly extends to the unrooted case with a few minor changes.

For each $S \subseteq V$, let $z_S \in \{0,1\}$ be a variable representing whether or not the vertices in S are the ones on which the tour is constructed; for each edge $e \in E$, we let $x_e \in \mathbb{Z}^+$ be a variable representing how many copies of e to include in the tour. Then, the following is a linear programming relaxation for the rooted budgeted prize-collecting traveling salesman problem.

$$\begin{aligned}
\text{maximize} \quad & \sum_{S \subseteq V : r \in S} |S| z_S \\
\text{subject to} \quad & \sum_{e : e \in \delta(S)} x_e \geq 2 \sum_{T : S \subsetneq T} z_T \quad \forall S \subsetneq V \\
& \sum_{e \in E} c_e x_e \leq D \\
& \sum_{S \subseteq V} z_S \leq 1 \\
& z_S, x_e \geq 0
\end{aligned}$$

The first constraint states that if the tour visits the vertices in a subset T and $S \subsetneq T$ then we must have at least two edges across the cut S. The dual of this linear program is given by the following.

$$\begin{aligned}
\text{minimize} \quad & \lambda_1 D + \lambda_2 \\
\text{subject to} \quad & (2 \sum_{T : T \subsetneq S} y_T) + \lambda_2 \geq |S| \quad \forall S \subseteq V : r \in S \\
& (2 \sum_{T : T \subsetneq S} y_T) + \lambda_2 \geq 0 \quad \forall S \subseteq V : r \notin S \\
& \sum_{S : e \in \delta(S)} y_S \leq \lambda_1 c_e \quad \forall e \in E \\
& \lambda_1, \lambda_2, y_S \geq 0
\end{aligned}$$

In order to construct a tour, we rely on a primal-dual subroutine that returns a tree containing the root. We first note that if we find $\lambda_1 \geq 0$ and $y_S \geq 0$ that satisfy the dual constraint for every edge, then we can always set λ_2 to be the maximum of 0 and $\max_{S \subseteq V : r \in S} \left[|S| - (2 \sum_{T : T \subsetneq S} y_T) \right]$ so that we have a feasible dual solution. Suppose that we first set the value of λ_1. The primal-dual subroutine uses this value to construct a full dual solution and corresponding tree containing r. Doubling this tree to form a tour may or may not be feasible with respect to the budget constraint. In particular, small values of λ_1 will produce tours that violate the budget and large values of λ_1 will produce tours that are far from optimal. Therefore, we need to adjust λ_1 to find a feasible solution with bounded approximation ratio. We first describe the primal-dual subroutine before presenting our overall algorithm.

3. Primal-Dual Subroutine The primal-dual subroutine for fixed λ_1 returns a tree containing the root, and is similar to the 2-approximation algorithm for the prize-collecting traveling salesman problem without a budget constraint presented by Goemans and Williamson [15]. Similar to Goemans and Williamson, we define a potential of a set S as a function of the dual variables of the strict subsets of S.

DEFINITION 1. For any subset $S \subseteq V$, we define the **potential** of S to be

$$\pi(S) := |S| - (2 \sum_{T : T \subsetneq S} y_T).$$

DEFINITION 2. A subset $S \subseteq V$ is **neutral** if $2 \sum_{T : T \subseteq S} y_T = |S|$. In other words, if $y_S = \frac{1}{2} \pi(S)$.

At the beginning of the primal-dual subroutine, we set all y_S to be 0 and set the collection of active sets to be all singleton nodes. Further, we set $T = \emptyset$. Then, in each iteration, we increase y_S corresponding to all $S \subseteq V$ in the collection of active sets until either a dual constraint for an edge between two sets becomes tight, or a set becomes neutral. If an edge becomes tight between two subsets S_1 and S_2, we add the edge to T and replace both S_1 and S_2 in the collection of active sets by $S_1 \cup S_2$. We remark that the potential of this new set S is then equal to

$$\pi(S) = \pi(S_1) + \pi(S_2) - 2y_{S_1} - 2y_{S_2}.$$

If instead a set becomes neutral, we mark it inactive and remove it from the collection of active sets. Once there are no more active sets, we prune inactive sets of degree 0 or 1 that do not contain the root and return the remaining set of edges T' (see Algorithm 1). Pruned subsets of degree 0 correspond to components that did not reach the root.

Algorithm 1 Primal-Dual Algorithm (PD(λ_1))

procedure PD($\lambda_1 \geq 0$)
  $y_S \leftarrow 0$, $T \leftarrow \{\}$.
  mark all $i \in V$ as active.
  while there exists an active subset **do**
    raise y_S uniformly for all active subsets S until either
    if an active set S becomes neutral **then**
      mark S as inactive.
    else if the dual constraint for edge e between S_1 and S_2 becomes tight **then**
      $T \leftarrow T \cup \{e\}$.
      mark $S = S_1 \cup S_2$ as active, remove S_1 and S_2 from the active subsets.
    end if
  end while
  $T' \leftarrow T$.
  while there exists a set S marked inactive such that $|\delta(S) \cap T'| < 2$ and $r \notin S$ **do**
    remove all edges with at least one endpoint in S from T'.
  end while
  return T'
end procedure

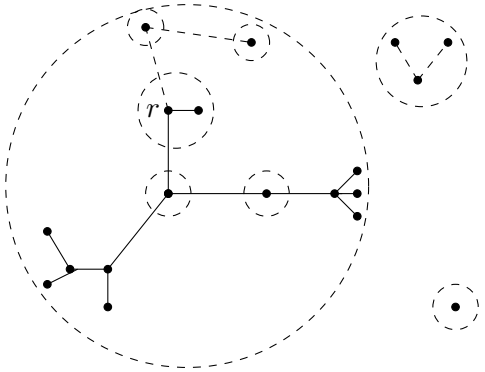


FIGURE 1. Pruning the forest T to find the tree T' containing r. Dashed circles represent subsets that were marked inactive, and edges that are pruned are dashed.

Since sets are removed from the set of active subsets when an incident edge becomes tight, the dual constraint for every edge will remain satisfied throughout the subroutine. Thus, λ_1 and y satisfy the dual constraints for every edge and we can extend them to a feasible dual solution $(y, \lambda_1, \lambda_2)$. Further, by construction, the dual constraint of every $e \in T$ is tight and the edges in the set T form a forest throughout the subroutine. This implies that, after the pruning phrase, T' is a tree containing r. Finally, beyond being part of a feasible solution, the construction of y also guarantees that $\mathcal{S}$, defined to be the collection of all sets active in some iteration throughout the subroutine, is laminar. This includes sets that are active even for a period of length zero (i.e. $y_S = 0$). Indeed, each set in $\mathcal{S}$, other than singletons, is the union of two disjoint other sets in $\mathcal{S}$. We define $\mathcal{S}^+ = \mathcal{S} \cup \{V\}$, which is also laminar.

4. Main Result In this section, we give an outline of how the parameterized primal-dual algorithm uses the primal-dual subroutine described in the previous section to construct a feasible tour with bounded approximation guarantee. Assume that we have run the primal-dual subroutine with value λ_1 to find a feasible dual solution $(y, \lambda_1, \lambda_2)$, where we may not know the actual value of λ_2. The subroutine also returns a tree of tight edges containing r. Our algorithm adjusts λ_1 so that this tree has cost very close to $\frac{1}{2}D$. In particular, the algorithm searches over possible values of λ_1 to find a threshold value such that $\text{PD}(\lambda_1^-)$ returns T' with cost $\geq \frac{1}{2}D$ and $\text{PD}(\lambda_1^+)$ returns T' with cost $< \frac{1}{2}D$. Here, $x^- = x - \varepsilon$ and $x^+ = x + \varepsilon$, where ε is arbitrarily small. Last, the algorithm prunes T'=$\text{PD}(\lambda_1^-)$ until doubling the resulting tree is a feasible solution. See Algorithm 2.

Algorithm 2 Overview of the Algorithm

1: Find λ_1 such that $\text{PD}(\lambda_1^-)$ returns T' with cost $\geq \frac{1}{2}D$ and $\text{PD}(\lambda_1^+)$ returns T' with cost $< \frac{1}{2}D$. See Section 6.
2: Prune $T' = \text{PD}(\lambda_1^-)$ to form a tree T_A with cost $\leq \frac{1}{2}D$ such that either T_A has cost $= \frac{1}{2}D$ or there exists an edge $e \in T^-$ such that $T_A \cup \{e\}$ has cost $> \frac{1}{2}D$. See Sections 6 and 7.
3: Double T_A and return the shortcutted tour.

The potentials outputted by the subroutine $\text{PD}(\lambda_1)$ help us give an upper bound on the size of an optimal solution and a lower bound on the size of our returned solution in order to provide an overall approximation guarantee. Let $O^\star$ be the subset of vertices visited by an optimal tour and $F^\star$ be the edges in that tour. Further, let O be the minimal set in $\mathcal{S}^+$ that contains $O^\star$. Since $V \in \mathcal{S}^+$, such a set always exists. In Section 5, we prove the following bound on the size of $O^\star$.

THEOREM 1.

$$|O^\star| \leq \lambda_1 D + \pi(O).$$

In Sections 6 and 7, we explicitly show how to find the threshold value λ_1 and how to construct the tree T_A of tight edges on a subset S_A such that $r \in S_A$ and cost $\sum_{e \in T_A} c_e \leq \frac{1}{2}D$. Doubling the edges in T_A and shortcutting produces a tour on S_A that has length at most D. Further, if the cost of T_A is $< \frac{1}{2}D$, we show that we can find an incident edge $\bar{e} \in T$ such that adding $\bar{e}$ to T_A creates a tree $\bar{T}$ on a subset $\bar{S}$ with cost $\sum_{e \in \bar{T}} c_e > \frac{1}{2}D$. In other words, adding one more edge to T_A makes doubling it infeasible. If instead the cost of T_A is equal to $\frac{1}{2}D$, then we simply let $\bar{T} = T_A$ and $\bar{S} = S_A$. Let Q be the maximum potential set in $\mathcal{S}^+$ containing $\bar{S}$. In Section 7, we prove the following bound on $|S_A|$.

THEOREM 2.

$$|S_A| > \frac{1}{2}\lambda_1 D + \pi(Q) - 1.$$

Thus, the parameterized primal-dual algorithm can find a feasible tour on a subset S_A where $|S_A|$ is bounded using the potential $\pi(Q)$. If $\pi(Q)$ is large, this implies that we have found a large subset our tour can visit. In particular, if $\pi(Q) \geq \lambda_2$, then $|S_A| \geq \frac{1}{2}\lambda_1 D + \lambda_2 \geq |O^*|$, by the feasibility of the dual solution. However, the value of λ_2 is unknown. Therefore, we use the potentials to bound the solution. In particular, suppose that $\pi(Q) \geq \pi(O)$, then

$$|S_A| + 1 > \frac{1}{2}[\lambda_1 D + \pi(O)] \geq \frac{1}{2}|O^\star|.$$

We may assume without loss of generality that $|O^\star|$ is even, since we can always make a copy of each vertex that has an edge of cost zero incident to the original. Hence, the above implies that if $\pi(Q) \geq \pi(O)$, then $|S_A| \geq \frac{1}{2}|O^\star|$.

However, it may be the case that $\pi(Q) < \pi(O)$. Therefore, the parameterized primal-dual algorithm finds the maximal set in $\mathcal{S}^+$ with potential strictly greater than $\pi(Q)$ that contains the root r and recurses on the graph induced by this set, returning the largest feasible tour found. If $\pi(Q) < \pi(O)$, then O must be a subset of the set we recurse on. Since the subset on which we recurse does not contain $\bar{S}$, by definition of Q as the maximum potential set containing $\bar{S}$, we recurse on a strict subgraph of G, implying that we eventually reach the case that $\pi(O) \leq \pi(Q)$ and $|S_A| \geq \frac{1}{2}O^\star$, yielding the 2-approximation guarantee. Finally, since we recurse on a strict subgraph of G, we call the subroutine at most $O(n)$ times.

THEOREM 3. *The parameterized primal-dual algorithm is a 2-approximation for the rooted budgeted prize-collecting traveling salesman problem.*

5. Upper Bound In this section, we provide the proof of Theorem 1 for any fixed λ_1. Recall that O is the minimal set in $\mathcal{S}^+$ that contains the optimal subset $O^\star$ and $F^\star$ is an optimal tour on $O^\star$. Assume that we have run the primal-dual subroutine with value λ_1 to obtain a dual solution $(y, \lambda_1, \lambda_2)$. Theorem 1 states that $|O^*| \leq \lambda_1 D + \pi(O)$. The proof relies on the feasibility of the dual solution along with the following lemma.

LEMMA 1. *For any $S \subseteq V$, $(2\sum_{T:T\subseteq S} y_T) \leq |S|$.*

Proof. Any set S can be partitioned into maximal disjoint laminar subsets $S_1, S_2, \ldots, S_c \in \mathcal{S}$. Therefore,

$$2\sum_{T:T\subseteq S} y_T = 2\sum_{i=1}^{c}\sum_{T:T\subseteq S_i} y_T \leq \sum_{i=1}^{c}|S_i| = |S|,$$

where the inequality comes from the fact that neutral subsets are marked inactive. □

Proof of Theorem 1 We first note that we can partition the powerset of O into subsets of $O - O^\star$ and subsets that contain vertices in $O^\star$.

$$2\sum_{T:T\subsetneq O} y_T = 2\sum_{T:T\subseteq O-O^\star} y_T + 2\sum_{\substack{T:T\subsetneq O\\ T\cap O^\star\neq\emptyset}} y_T. \tag{1}$$

Thus,

$$\begin{aligned}|O| &= 2\sum_{T:T\subsetneq O} y_T + \pi(O)\\ &= 2\sum_{T:T\subseteq O-O_\star} y_T + 2\sum_{\substack{T:T\subsetneq O\\ T\cap O^\star\neq\emptyset}} y_T + \pi(O)\\ &\leq |O-O^\star| + 2\sum_{\substack{T:T\subsetneq O\\ T\cap O^\star\neq\emptyset}} y_T + \pi(O),\end{aligned}$$

where the first line holds by the definition of potentials, the second by Equation (1), and the last line by Lemma 1. Rearranging, we get

$$\begin{aligned}|O^\star| &\le 2 \sum_{\substack{T:T\subseteq O\\ T\cap O^\star\neq\emptyset}} y_T + \pi(O)\\ &\le \sum_{e\in F^\star}\sum_{T:e\in\delta(T)} y_T + \pi(O)\\ &\le \lambda_1 \sum_{e\in F^\star} c_e + \pi(O)\\ &\le \lambda_1 D + \pi(O).\end{aligned}$$

The second line holds since $F^\star$ is a tour on $O^\star$ and so, by the minimality of O, each subset T with $y_T > 0$ that contains a subset of $O^\star$ has at least two edges in its cut $\delta(T)$. The third line holds by the dual feasibility of $(y, \lambda_1, \lambda_2)$, and the last line holds by the primal feasibility of $F^\star$. □

6. Setting λ_1 We now turn our attention to finding λ_1 and constructing a feasible tour. Our goal is to set λ_1 for the primal-dual subroutine so as to find a tree containing the root with cost very close to $\frac{1}{2}D$ and such that the set of spanned vertices has high potential. Note that λ_1 controls the cost of the edges, and as λ_1 increases, edges become more expensive yielding smaller connected components in the primal-dual subroutine. In particular, for $\lambda_1 = 0$ all edges go tight immediately and for $\lambda_1 > n/(2\min_{e:c_e>0} c_e)$ all vertices go neutral before a single non-zero edge goes tight. If edges go tight and/or subsets go neutral at the same time, we break ties in some order. As discussed later, this ordering is important since it changes the outputted tree T'.

If a minimum spanning tree on G has cost $\le \frac{1}{2}D$, then we double this tree to get a feasible and optimal tour. Otherwise, suppose that we have found values λ_l and λ_r ($\lambda_l < \lambda_r$) such that $\mathrm{PD}(\lambda_l^+)$ returns T' with cost $\ge \frac{1}{2}D$ and $\mathrm{PD}(\lambda_r^-)$ returns T' with cost $< \frac{1}{2}D$. Here, $x^- = x - \varepsilon$ and $x^+ = x + \varepsilon$, where ε is arbitrarily small.

Lemma 2. *In polynomial time, we can find a threshold value λ_1 such that PD(λ_1^-) returns T' with cost $\ge \frac{1}{2}D$ and PD(λ_1^+) returns T' with cost $< \frac{1}{2}D$.*

Proof. We refer to an edge going tight during the primal-dual subroutine as an edge event and we refer to a subset going neutral as a subset event. We prove the result by induction. Assume we have values λ_l and λ_r such that, for some k, the first $k \ge 0$ events are the same when running the subroutine for any λ_1 between λ_l^+ and λ_r^-. Further, assume that for each subset S we can find values α_S and β_S such that at the end of the first k events $y_S = \lambda_1\alpha_S + \beta_S$ for any λ_1 between λ_l^+ and λ_r^-. Note that this is trivially true for the base case with λ_l and λ_r defined above and $k = 0$ since all y values will be zero.

To find the next event to occur, we need to find the time after the kth event that each subset will go neutral and each edge will go tight. Observe that an active set S will go neutral at time

$$\frac{1}{2}|S| - \sum_{T\subseteq S} y_T = \frac{1}{2}|S| - \sum_{T\subseteq S}[\lambda_1\alpha_T + \beta_T],$$

an edge with exactly one endpoint in an active component will go tight at time

$$\lambda_1 c_e - \sum_{T:e\in\delta(T)} y_T = \lambda_1 c_e - \sum_{T:e\in\delta(T)} [\lambda_1\alpha_T + \beta_T],$$

and an edge with both endpoints in different active components will go tight at time $\frac{1}{2}$ the above amount. The minimum of these values determines the next event to occur. Since all of these times

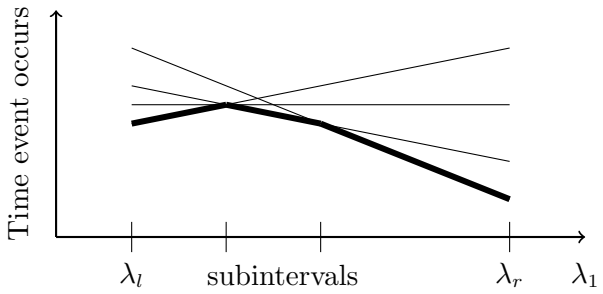


FIGURE 2. Finding the subintervals between λ_l and λ_r where the time of the next event is in bold.

are affine in λ_1, we can divide the interval between λ_l^+ and λ_r^- into smaller subintervals such that the first $k+1$ events are identical on these subintervals. Note that if more than two events tie, some events are *degenerate* in that those events do not occur next in any subinterval. See Figure 2.

Considering these subintervals, we either identify a threshold point λ_1 or we identify a subinterval between $\hat{\lambda}_l$ and $\hat{\lambda}_r$ such that $\text{PD}(\hat{\lambda}_l^+)$ returns T' with cost $\geq \frac{1}{2}D$ and $\text{PD}(\hat{\lambda}_r^-)$ returns T' with cost $< \frac{1}{2}D$. Further, since the time of the $(k+1)$th event is an affine function in λ_1, we can add this function to the affine function $y(S)$ for each active set S to get the new affine function for this y value, updating the α's and β's accordingly. Thus, the inductive hypothesis holds and eventually we find such a threshold point. □

We use this threshold point λ_1 to understand the subroutine $\text{PD}(\lambda_1)$. In particular, we will show that if the subroutine breaks event ties in one way, then the returned tree is the same as $\text{PD}(\lambda_1^-)$, and if the subroutine breaks ties another way, then the resulting tree is the same as $\text{PD}(\lambda_1^+)$. Consider running the subroutine for λ_1^+ and λ_1^- and comparing event by event. We let y^+ correspond to the y variables when running $\text{PD}(\lambda_1^+)$ and let y^- correspond to the y variables when running $\text{PD}(\lambda_1^-)$. The following lemma shows that the two routines look near-identical.

LEMMA 3. *Throughout the two subroutines, the following two properties hold:*
- *All active components in (V,T) are the same except for during infinitesimal time.*
- *For all $S \subseteq V$, the difference between y_S^+ and y_S^- is infinitesimal.*

Here, infinitesimal means that the time or difference goes to zero as $\epsilon \to 0$.

Proof. At the start of the subroutines this is true since all y^+ and y^- variables are zero. Now assume that it is true at some time t into the subroutines. As argued above, the next event to occur depends on the minimum of functions linear in λ_1. Further, since the current active components are the same, the possible subset and edge events are the same.

In particular, the time for each subset to go neutral in $\text{PD}(\lambda_1^+)$ is $\frac{1}{2}|S| - \sum_{T \subseteq S} y_T^+$, which is infinitesimally different from the time for that subset to go neutral in $\text{PD}(\lambda_1^-)$. Similarly, the time for each edge to go tight is at most infinitesimally different between the two subroutines. Therefore, the next event to occur is only different between the two subroutines if two events occur at the same time for $\text{PD}(\lambda_1)$.

If the next event is the same for the two subroutines, then the active components remain the same and we raise all active components by amounts differing by an infinitesimally different amount. Therefore, the inductive properties continues to hold. Otherwise, suppose the next event is different. We consider four cases:

1. Subset X goes neutral for $\text{PD}(\lambda_1^-)$ and subset Y goes neutral for $\text{PD}(\lambda_1^+)$.
2. Edge e goes tight for $\text{PD}(\lambda_1^-)$ and edge f goes tight for $\text{PD}(\lambda_1^+)$.
3. Edge e goes tight for $\text{PD}(\lambda_1^-)$ and subset X goes neutral for $\text{PD}(\lambda_1^+)$.
4. Subset X goes neutral for $\text{PD}(\lambda_1^-)$ and edge e goes tight for $\text{PD}(\lambda_1^+)$.

In the first case, the times for both X and Y to go neutral must be infinitesimally different and the other subset goes neutral immediately after the first. Therefore, once both X and Y are neutral, the difference of the amounts by which we have raised the y variables in $\text{PD}(\lambda_1^-)$ and in $\text{PD}(\lambda_1^+)$ is infinitesimally small and the current active components are the same. Thus, the two inductive properties continue to hold.

Similarly for the second case, if e and f are not between the same two components, the other edge goes tight immediately after, and the inductive properties continue to hold. Otherwise, e and f are between the same components. Thus, when e goes tight, f is no longer eligible to go tight but the newly merged active component is the same for both subroutines. Again, the inductive properties continue to hold.

In the third case, if edge e has an endpoint in an active component that is not X, then e goes tight immediately after X goes neutral for $\text{PD}(\lambda_1^+)$ and the components remain the same, maintaining the inductive properties. Otherwise, one endpoint of e must be in X and the other endpoint of e is in an inactive component Y, and right after e goes tight for $\text{PD}(\lambda_1^-)$, the newly merged subset has infinitesimally little potential left and goes inactive immediately thereafter. Further, if in the future any active component merges with X or Y in $\text{PD}(\lambda_1^+)$ then the edge e will immediately go tight after, maintaining that the active components stay the same. Again, this maintains the inductive properties.

Lastly, note that the time for a subset to go neutral has a negative slope in λ_1 and the time for an edge to go tight has a positive slope in λ_1. Since $\lambda_1^+ > \lambda_1^-$ and the y variables vary by an infinitesimally small amount, the fourth case cannot occur. In all cases, the inductive properties continue to hold, and the lemma holds. □

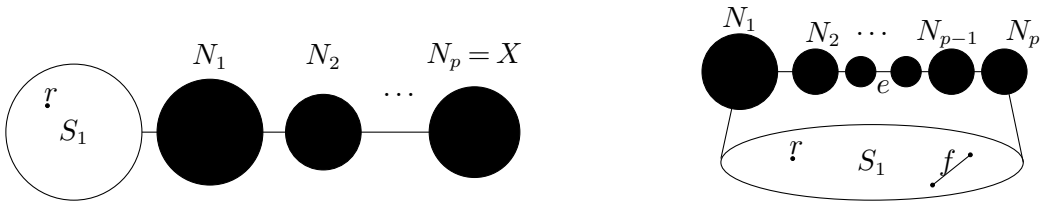
The proof of Lemma 3 exactly exhibits the differences between the two subroutines. First, there may be subsets that are neutral and marked inactive in $\text{PD}(\lambda_1^+)$ but have infinitesimally small potential in $\text{PD}(\lambda_1^-)$. Second, there may be pairs of edges that went tight between the same components. Lastly, there may be edges in $\text{PD}(\lambda_1^-)$ that do not exist in $\text{PD}(\lambda_1^+)$. However, these edges are between inactive components and components with infinitesimally small potential such that the merged component immediately goes inactive and that component remained untouched for the rest of the subroutine. Therefore, these edges are not pruned in $\text{PD}(\lambda_1^-)$ if and only if the inactive component contains the root.

Assume that we run $\text{PD}(\lambda_1)$ breaking event ties to behave the same as $\text{PD}(\lambda_1^+)$. Then, $\text{PD}(\lambda_1)$ returns T' with cost $< \frac{1}{2}D$. However, we can think about reversing these ties one by one. In particular, consider breaking the first i ties according to $\text{PD}(\lambda_1^-)$ and then the rest by $\text{PD}(\lambda_1^+)$. By the analysis in Lemma 3, reversing these ties only changes the y variables by an infinitesimally small amount. The only difference occurs when entering the pruning phrase.

Thus, eventually we find the smallest i such that breaking the first i ties according to $\text{PD}(\lambda_1^-)$ returns T' with cost $\geq \frac{1}{2}D$. In other words, we have either identified a neutral subset X such that $r \notin X$ and marking X active rather than inactive changes T' to have cost $\geq \frac{1}{2}D$ or we have identified two edges e and f that tie such that adding e instead of f changes T' to have cost $\geq \frac{1}{2}D$. From here on, we assume that we always run $\text{PD}(\lambda_1)$ according to these tie-breaking rules.

7. Constructing a Tour Continuing from the previous section, let y be the dual variables at the end of running $\text{PD}(\lambda_1)$, let T' be the set of edges after the pruning phase, and let $\mathcal{S}$ be defined as before. Lastly, let $\pi(S)$ be the potential of $S \subseteq V$ given y. By construction, the tree returned by $\text{PD}(\lambda_1)$ has cost $\geq \frac{1}{2}D$. Recall from Section 6 that either

1. there exists a neutral subset $X \in \mathcal{S}$ such that $r \notin X$ and if X is marked inactive then the pruned tree would have cost $< \frac{1}{2}D$ or
2. there exist tight edges $e \in T$ and $f \notin T$ such that if we swap e with f in T then the pruned tree would have cost $< \frac{1}{2}D$.



(a) Case 1: Marking X as inactive. (b) Case 2: Replacing e with f.

FIGURE 3. Neutral subsets pruned in each case to yield component S_1 with cost $< \frac{1}{2}D$. The subset S_1 may itself be neutral but not pruned.

In the first case, when X is marked inactive, then a path of neutral subsets $N_1, N_2, \ldots, N_p = X$ is pruned yielding a component S_1 containing the root r with cost $< \frac{1}{2}D$. Similarly, in the second case, edge e prevented some neutral subsets $N_1, N_2, \ldots, N_p$ from being pruned that had degree > 1. However, by removing e and replacing it with f, these subsets are pruned and we are left with component S_1 containing r with cost $< \frac{1}{2}D$. See Figures 3a and 3b.

For both cases, we use this threshold event to produce a tree T_A on a subset of vertices S_A with $r \in S_A$ and cost $\leq \frac{1}{2}D$. In doing so, we also find another tree $\bar{T}$ on a subset of vertices $\bar{S}$ containing S_A of cost $\geq \frac{1}{2}D$ such that $|S_A| \geq |\bar{S}| - 1$. Then, doubling T_A yields a feasible tour F_A that visits at most one node less than there are in $\bar{S}$. The tree $\bar{T}$ will be helpful later in obtaining a lower bound for $|S_A|$.

We start by setting T_A to be the edges in T' that span S_1. By construction, these edges have cost $< \frac{1}{2}D$. We then try to grow T_A as much as possible along the path from S_1 to $N_1, N_2, \ldots, N_p$. First, suppose that we can add this full path and the edges in T' that span each N_i to T_A without exceeding cost $\frac{1}{2}D$. Then, we set T_A to be this expanded tree and $S_A = S_1 \cup N_1 \cup \ldots \cup N_p$. In other words, T_A is equal to T' at the end of PD(λ_1). Further, we set $\bar{T} = T_A$. By construction, the cost of $\bar{T}$ is $= \frac{1}{2}D$ and $|S_A| = |\bar{S}|$.

Otherwise, we continue to add N_1, N_2, ... to our tree until we reach a set $\bar{X} \in \{N_1, N_2, \ldots, N_p\}$ such that adding the edges in T' that span $\bar{X}$ in T' to T_A implies that $\sum_{e \in T_A} c_e > \frac{1}{2}D$. In other words, we cannot add this whole subset to our tree without going over budget. Let $e = (u, v)$ be the edge that connects $\bar{X}$ to T_A in T'. If adding e to T_A already brings the cost of T_A strictly over $\frac{1}{2}D$, then we stop growing T_A and set $\bar{T} = T_A \cup \{e\}$. Otherwise, we add e to T_A and run a procedure **pick**$(\bar{X}, v, T_A)$ that picks a subset of the edges in T' spanning $\bar{X}$ including v.

Specifically, the procedure **pick**(X, w, T_A) adds to T_A a set of edges in T' that span a subset of component X including w. We denote by $X_1, X_2 \in \mathcal{S}$ the two components that merged to form X and by $e' = (u, v)$ the edge that connects X_1 and X_2 in T'. Without loss of generality, $u, w \in X_1, v \in X_2$. Further, let T'_1 and T'_2 be the edges in T' with both endpoints in X_1 and X_2, respectively. See Figure 4.

If the total cost of edges in $T_A \cup T'_1$ is greater than $\frac{1}{2}D$, then we know we should only add edges in this subtree to T_A and we recursively invoke **pick**(X_1, w, T_A). If instead the total cost of edges in $T_A \cup T'_1 \cup \{e'\}$ is less than $\frac{1}{2}D$, then we can feasibly add e' and all edges in T'_1 without violating the budget. Thus, the procedure adds all these edges to T_A and recursively invokes **pick**(X_2, v, T_A) to pick the remaining edges in T'_2. Finally, if the cost of edges in $T_A \cup T'_1$ is less than or equal to $\frac{1}{2}D$, but greater than $\frac{1}{2}D - c_{e'}$, then we cannot quite make it to T'_2 without going over budget. In this case, the procedure adds all edges in T'_1 to T_A and sets $\bar{T} = T_A \cup \{e'\}$. See Algorithm 3.

At the end of the procedure, we produce a tree T_A of cost $\leq \frac{1}{2}D$ that spans a subset S_A with $r \in S_A$ along with a tree $\bar{T}$ of cost $\geq \frac{1}{2}D$ that spans a subset $\bar{S}$ where $|\bar{S}| \leq |S_A| + 1$. Further, if $|\bar{S}| = |S_A| + 1$, then $\bar{T}$ has cost $> \frac{1}{2}D$. We will use $\bar{T}$ to prove a bound on $|\bar{S}|$, which in turn will give a bound on $|S_A|$.

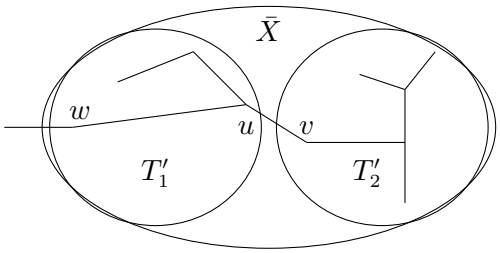


FIGURE 4. Illustration of the pick procedure.

Algorithm 3 Pick Routine (**pick**(X, w, T_A))

procedure pick(X, w, T_A)
  let X_1, X_2 be the two components that merged on edge $e' = (u, v)$ to form X
  (w.l.o.g. $w, u \in X_1$ and $v \in X_2$)
  let T_i' be the edges in T' such that both endpoints are in X_i for $i = 1, 2$
  if the cost of $T_A \cup T_1'$ is $\geq \frac{1}{2}D$ **then**
    call **pick**(X_1, w, T_A)
  else if the cost of $T_A \cup T_1' \cup \{e'\}$ is $< \frac{1}{2}D$ **then**
    $T_A = T_A \cup T_1' \cup \{e'\}$, and call **pick**$(X_2, w, T_A)$
  else if the cost of $T_A \cup T_1' \cup \{e'\}$ is $> \frac{1}{2}D$ **then**
    $T_A = T_A \cup T_1'$, $\bar{T} = T_A \cup \{e\}$
  else
    $T_A = T_A \cup T_1' \cup \{e\}$, $\bar{T} = T_A$
  end if
end procedure

Let $Q \in \mathcal{S}^+$ be the maximum potential subset containing $\bar{S}$. Our goal is to prove Theorem 2, which states that

$$|S_A| \geq \frac{1}{2}\lambda_1 D + \pi(Q) - 1.$$

Our proof relies, informally speaking, on the vertex $\bar{v}$ at which the pick routine stopped. We define $\bar{v}$ as the (unique) vertex in $\bar{S}$ that has fewer edges incident in $\bar{T}$ than it does in T'; though many vertices in the graph have fewer incident edges in $\bar{T}$ than in T', the definition of the pick routine, combined with the characterization in Figures 3a and 3b guarantees that only one of them is in $\bar{S}$. Further, if $|\bar{S}| > |S_A|$, then $\bar{v}$ is exactly the one node in $\bar{S}$ that is not in S_A. Before proving Theorem 2, we first state the following useful lemma, the proof of which we delay to the end of the section as it closely closely resembles that of Goemans and Williamson [15] for the Prize-Collecting Steiner Tree Problem.

LEMMA 4.

$$\sum_{e \in \bar{T}} \sum_{S : e \in \delta(S)} y_S \leq 2 \sum_{\substack{T : T \cap \bar{S} \neq \emptyset \\ \bar{v} \notin T}} y_T. \tag{2}$$

Proof of Theorem 2. By construction of $\bar{T}$, vertices in $Q - \bar{S}$ are either (i) in a single set $N \subset Q - \bar{S}$ that denotes the union of pruned subsets and sets N_i in Q that were not reached by the pick procedure (cf. Figure 1) or (ii) in the set $\bar{X} \in \{N_1, N_2, \ldots, N_p\}$ set on which we started our pick routine. Thus, we may partition $Q = N \cup (\bar{X} - \bar{S}) \cup \bar{S}$. Notice that N is a union of neutral subsets and thus is itself neutral, that is,

$$|N| = 2 \sum_{T : T \in \mathcal{S}_N} y_T.$$

Similarly, let $\mathcal{S}_X$ be all subsets in $\mathcal{S}$ that are subsets of $\bar{X}$ and contain vertices in $\bar{X}-\bar{S}$. We may partition the subsets of $\bar{X}$ into sets that contain nodes in $\bar{X}-\bar{S}$ and sets that do not; we then find (since $\bar{X}$ is neutral)

$$|\bar{X}| = 2\sum_{T:T\in\mathcal{S}_X} y_T + 2\sum_{T:T\subseteq\bar{X}\cap\bar{S}} y_T \le 2\sum_{T:T\in\mathcal{S}_X} y_T + |\bar{X}\cap\bar{S}|$$

where the inequality follows by Lemma 1. Thus, $|\bar{X}-\bar{S}| \le 2\sum_{T:T\in\mathcal{S}_X} y_T$.

The definition of $\bar{v}$ guarantees that any subset in $\mathcal{S}$ that contains vertices in $\bar{S}$ and $\bar{X}-\bar{S}$ also contains $\bar{v}$. Therefore, subsets T that intersect with $\bar{S}$ but do not contain $\bar{v}$ are neither in $\mathcal{S}_N$ nor in $\mathcal{S}_X$. It follows that

$$\begin{aligned}
|Q| &= 2\sum_{T:T\subseteq Q} y_T + \pi(Q)\\
&\ge 2\sum_{\substack{T:T\cap\bar{S}\neq\emptyset\\ \bar{v}\notin T}} y_T + 2\sum_{T:T\in\mathcal{S}_N} y_T + 2\sum_{T:T\in\mathcal{S}_X} y_T + \pi(Q)\\
&\ge 2\sum_{\substack{T:T\cap\bar{S}\neq\emptyset\\ \bar{v}\notin T}} y_T + |N| + |\bar{X}-\bar{S}| + \pi(Q)\\
&= 2\sum_{\substack{T:T\cap\bar{S}\neq\emptyset\\ \bar{v}\notin T}} y_T + |Q-\bar{S}| + \pi(Q)
\end{aligned}$$

Rearranging,

$$|\bar{S}| \ge 2\sum_{\substack{T:T\cap\bar{S}\neq\emptyset\\ \bar{v}\notin T}} y_T + \pi(Q) \ge \sum_{e\in\bar{T}}\sum_{S:e\in\delta(S)} y_S + \pi(Q) = \lambda_1\cdot\sum_{e\in\bar{T}} c_e + \pi(Q).$$

The second inequality follows from Lemma 4, whereas the final equality is due to the fact that the dual constraints are tight for all edges obtained by the primal-dual subroutine. If $|\bar{S}| = |S_A|$, this completes the proof of the theorem. Else, suppose that $|\bar{S}| = |S_A|+1$. Then, $\sum_{e\in\bar{T}} c_e > \frac{1}{2}D$, and the asserted inequality still holds. □

Proof of Lemma 4. Consider a single iteration of the algorithm, and let $\mathcal{C}$ be the current set of components C such that $|\delta(C)\cap\bar{T}| \ge 1$. In other words, these are the components incident to edges in $\bar{T}$. We can partition $\mathcal{C}$ into active components $\mathcal{C}_A$ and inactive components $\mathcal{C}_I$. Further, let $C_{\bar{v}}$ be the component that contains $\bar{v}$. We first show that active components are not incident to too many edges in $\bar{T}$ on average.

Starting with the graph $(V,\bar{T})$, we consider the graph obtained by shrinking each component in $\mathcal{C}$ to a single vertex and removing all vertices not in a component in $\mathcal{C}$. The remaining edges are a subset of $\bar{T}$ and form a tree on $\mathcal{C}$. Further, the degree of each vertex v in this tree is $d_v = |\delta(C)\cap\bar{T}|$, where C is the corresponding component. We set R to be the set of vertices corresponding to components in $\mathcal{C}_A$ and set B be the set of vertices corresponding to components in $\mathcal{C}_I$. Since the edges in $\bar{T}$ form a tree on $\mathcal{C}$,

$$\sum_{v\in R} d_v + \sum_{v\in B} d_v \le 2|R| + 2|B| - 2.$$

Let C be an inactive component such that $|\delta(C)\cap\bar{T}| = 1$ and $r\notin C$. Note that $|\delta(C)\cap T'| > 1$, where T' is the set of all tight edges returned by the subroutine, since otherwise C would have been pruned. Therefore, the other component incident to C was not chosen to be part of $\bar{T}$. This implies that either C is $\bar{X}$ or C is some X_1 encountered during the picking procedure such that we only chose vertices in X_1 and not X_2. In either case, $\bar{v}\in C$.

If $C_{\bar{v}} \in \mathcal{C}_I$, then this and the component containing the root are the only possible components in B with degree $=1$ and $\sum_{v\in B} d_v \geq 2|B|-2$. In other words,

$$\sum_{C\in\mathcal{C}_A} |\delta(C)\cap\bar{T}| = \sum_{v\in R} d_v \leq 2|R| = 2|\mathcal{C}_A| \tag{3}$$

Otherwise, $C_{\bar{v}} \in \mathcal{C}_A$ and the component containing the root is the only possible vertex in B with degree <2. Thus, $\sum_{v\in B} d_v \geq 2|B|-1$. This implies that

$$\sum_{C\in\mathcal{C}_A} |\delta(C)\cap\bar{T}| \leq 2|R|-1 = 2|\mathcal{C}_A|-1. \tag{4}$$

We now prove the main claim in Equation (2) using a proof by induction. At the start of the algorithm, the LHS and RHS are both equal to 0. On each iteration, let $\mathcal{C}_A$ be the set of active components C such that $|\delta(C)\cap\bar{T}| \geq 1$ and let $C_{\bar{v}}$ be the component containing $\bar{v}$. Suppose that we raise y_{C_i} by ε for every active component C_i. The LHS of the claim is raised by $\sum_{C\in\mathcal{C}_A} |\delta(C)\cap\bar{T}|\varepsilon$. If $C_{\bar{v}} \in \mathcal{C}_I$, then the RHS is raised by exactly $2|\mathcal{C}_A|\varepsilon$. By Equation (3), the LHS $\leq$ RHS. If, on the other hand, $C_{\bar{v}} \in \mathcal{C}_A$, then the RHS is raised by exactly $2(|\mathcal{C}_A|-1)\varepsilon$. By Equation (4), the LHS $\leq$ RHS. In both cases, the inductive statement continues to hold. □

8. Extensions In this section we extend the algorithm and its analysis in three different directions. First, we show that the 2-approximation extends to the version of the problem in which each vertex has a weight associated to it and the goal is to maximize the combined weight of the nodes that are part of the tour. Second, we show that our results also hold when the goal is to return a tree rather than a tour. Last, we show that the algorithm, in a natural way, also gives a 2-approximation for the unrooted problem. The usual approach to show that an approximation algorithm for a rooted problem also gives an approximation for the unrooted would be to iterate over all possible roots and take the best solution. Here, we show that the analysis for the rooted case also applies directly to the unrooted case.

Weights Suppose that each vertex v is associated with an integer weight $p_v \geq 0$ and the goal is to find a feasible tour F that visits a subset $S \subseteq V$ to maximize $\sum_{v\in S} p_v$. We can imagine transforming this problem to an equivalent unweighted version by creating copies of each vertex v with zero cost edges to v. Then in the primal-dual subroutine, all these added edges go tight instantaneously, yielding a weighted "cluster" with potential equal to p_v. Thus, we don't even need to perform the transformation and can actually just begin the algorithm with these weighted clusters as our initial active sets with potential equal to the weight of v. All proofs continue to hold through the equivalence to the unweighted version.

THEOREM 4. *The parameterized primal-dual algorithm is a 2-approximation for the rooted budgeted prize-collecting traveling salesman problem with weights.*

Trees Second, we discuss the extension of the algorithm to the rooted budgeted prize-collecting minimum spanning tree problem. The corresponding linear programming relaxation for the budgeted prize-collecting minimum spanning tree problem is given below.

$$\begin{aligned} \text{maximize} \quad & \sum_{S\subseteq V: r\in V} |S| z_S \\ \text{subject to} \quad & \sum_{e: e\in\delta(S)} x_e \geq \sum_{T: S\subsetneq T} z_T \quad \forall S \subsetneq V \end{aligned}$$

$$\sum_{e\in E} c_e x_e \le D$$
$$\sum_{S\subseteq V} z_S \le 1$$
$$z_S, x_e \ge 0$$

The only difference is the removal of the factor of 2 from the first constraint which changes the dual constraint associated with each subset S to become

$$\sum_{T:T\subsetneq S} y_T + \lambda_2 \ge |S| \quad \forall S \subseteq V : r \in S$$
$$\sum_{T:T\subsetneq S} y_T + \lambda_2 \ge 0 \quad \forall S \subseteq V : r \notin S.$$

Based on this change, we redefine the potential of a set to be

$$\pi(S) = |S| - \sum_{T:T\subsetneq S} y_T$$

and a set to be neutral if $\sum_{T:T\subseteq S} y_T = |S|$. Given these new definitions, the primal-dual subroutine runs exactly as it did before. Last, we set the threshold value λ_1 such that the tree returned by $\mathrm{PD}(\lambda_1^-)$ has cost $\ge D$ and the tree returned by $\mathrm{PD}(\lambda_1^+)$ has cost $< D$. We then run the pick procedure to find a tree just within budget D. It is easily verified that all proofs continue to hold for this variant.

THEOREM 5. *The parameterized primal-dual algorithm is a 2-approximation for the rooted budgeted prize-collecting minimum spanning tree problem.*

Unrooted The most intuitive way to think about the unrooted version is to think of considering every possible component in T as a possible tour, not just the component containing the root. With that interpretation, one can summarize our results for the unrooted case as follows: the linear program then captures the objective by summing over all subsets $S \subseteq V$ and the dual has the constraint $(2\sum_{T:T\subsetneq S} y_T) + \lambda_2 \ge |S|$ for every $S \subseteq V$, rather than just ones containing the root. The definition of potentials and neutral remain exactly the same and the only difference in the algorithm is that we prune all sets S that are marked inactive and have $|\delta(S) \cap T'| = 1$. Each remaining component corresponds to a potential tour. Finally, instead of recursing only on the subset containing the root, we now recurse on all maximal subsets with potential $> \pi(Q)$. Since $\mathcal{S}$ is a laminar family, we recurse on disjoint sets of vertices, so the set of graphs on which we recurse is also a laminar family and we call the subroutine $O(n)$ times.

The statement and proof of the upper bound in Theorem 1 and the lower bound in Theorem 2 remain unchanged. The statements of Lemma 3 and Lemma 4 also continue to hold true, but the proof of Lemma 4 changes a little bit. After establishing $\sum_{v\in R} d_v + \sum_{v\in B} d_v \le 2|R| + 2|B| - 2$, we let C be any inactive component such that $|\delta(C) \cap \bar{T}| = 1$. As before, we know that $\bar{v} \in C$. However, if $C_{\bar{v}} \in \mathcal{C}_I$, then $C_{\bar{v}}$ is the only possible component in B with degree 1 and $\sum_{v\in B} d_v \ge 2|B| - 1$; thus, we get the stronger inequality

$$\sum_{C\in\mathcal{C}_A} |\delta(C) \cap \bar{T}| = \sum_{v\in R} d_v \le 2|R| - 1 = 2|\mathcal{C}_A| - 1.$$

Similar, if instead $C_{\bar{v}} \in \mathcal{C}_A$, then all vertices in B have degree at least 2 and $\sum_{v\in B} d_v \ge 2|B|$ implying that

$$\sum_{C\in\mathcal{C}_A} |\delta(C) \cap \bar{T}| \le 2|R| - 2 = 2|\mathcal{C}_A| - 2,$$

which is again a stronger inequality. The rest of the proof remains the same.

9. Computational Experiments In this section, we complete computational experiments in order to better understand the performance of our algorithm in practice. The primal-dual algorithm as detailed in this paper was implemented in C++11 using binary search to find λ_1. The experiments were conducted on a Dell R620 with two Intel 2.70GHz 8-core processors and 96GB of RAM.

The first set of graphs we used for the experiments are the 37 symmetric TSP instances with at most 400 nodes in the TSPLIB data set [21]. The second set of graphs are 37 weighted instances constructed using the Citi Bike network of bikesharing stations in New York City. Each instance corresponds to a week of usage data at these stations, and the weight of a vertex corresponds to the number of broken docks at that station during that week. The number of broken docks was estimated from usage data with a probabilistic method similar to that of Kaspi, Raviv, and Tzur [18]. Details about both types of constructed instances are given in Table 1. Throughout this section, we only considered the unrooted case.

Instance Type	$\|V\|$	$\|E\|$	Total Vertex Weight
TSPLIB	158.14	15658.43	158.14
Bike	319.54	4634.77	1302.51

TABLE 1. Graph statistics for both groups of graphs averaged over all instances.

For each test graph G, we first found an upper bound on the cost of a tour by computing 2 times the cost of a minimum spanning tree in G. We then set the budget for our tour to be $f = 25\%$, 50%, or 75% of this upper bound. W denotes the total weight of the vertices; for TSPLIB instances, this is the number of vertices. After finding our solution of weight A, we compute an upper bound on the weight of visited vertices $U = \min(\lambda_1 D + \max_{S\in\mathcal{S}} \pi(S), W)$ and record the percent optimality gap as $100 \times (U - A)/U$. Results are given in Table 2. The column headed % Weight gives the percentage of the total weight W captured by the constructed tour, and the one headed % Budget gives the percentage of the distance budget used after shortcutting the tree.

Instance Type	f	Time (s)	# Recursions	% Opt. Gap	% Weight	% Budget
TSPLIB	0.25	74.16	0.59	46.67	33.06	77.38
TSPLIB	0.5	72.61	0.14	41.89	58.08	69.89
TSPLIB	0.75	71.24	0.22	18.62	81.38	68.80
Bike	0.25	25.15	0.28	45.74	43.37	66.90
Bike	0.5	33.21	0.28	25.89	74.01	67.13
Bike	0.75	30.46	0.05	8.29	91.68	67.37

TABLE 2. Computational results of the primal-dual algorithm for each group of graphs and budget with results averaged over all instances.

We report several interesting structural results. First, the average time seems to be heavily influenced by the number of edges; the bike instances were quicker to complete even though the average number of nodes was higher. However, the average time does not seem to grow with the budget (and hence with the size of the computed solution) since most of the time is spent finding λ_1. The average optimality gap, on the other hand, does improve with the budget. This is likely due to the fact that for larger budgets the upper bound is given by W rather than $\lambda_1 D + \max_{S\in\mathcal{S}} \pi(S)$. Also of interest is that $\max_{S\in\mathcal{S}} \pi(S)$ contributed little to our upper bound U. As a result, our optimality gaps depend mostly on the value of λ_1, rather than the potentials, and may be far from tight. However, the fact that on average we only use around $2/3$ of the distance budget implies that the solutions could be improved as well. To ensure that we use a larger part of the budget, we ran further experiments on the Citi Bike instances; in these, we ran binary search over possible

virtual budgets in the input until finding one for which the resulting tour uses at least 90% of the actual budget. This significantly reduced our optimality gaps as shown in Table 3.

f	0.25	0.5	0.75
% Opt. Gap without Virtual Budgets	45.74%	25.89%	8.29%
% Opt. Gap with Virtual Budgets	27.96%	11.87%	0.17%

TABLE 3. Improvement of Optimality Gap using Virtual Budgets.

10. Conclusion and Future Work In this paper, we provide 2-approximation algorithms for the rooted and unrooted budgeted prize-collecting traveling salesman and the prize-collecting minimum spanning tree problem that have at their base a classic primal-dual approach. The key insights are to use constructed potentials to evaluate feasible subsets of vertices to visit and to identify the structure of a good tour. In particular, we construct a tree that closely follows the structure of the laminar collection of subsets with positive dual value. Further, we ensure this tree is just within budget in that adding one extra edge, and doubling the tree in the case of the tour, violates the budget.

An obvious open question seeks to improve the approximation guarantee or prove the current guarantee is the best possible. Specifically, it would be interesting to know whether or not a (3/2)-approximation algorithm is possible given that that is the current best guarantee for the unconstrained traveling salesman problem. Another interesting direction is to investigate whether the number of recursions, currently bounded by $O(n)$, can be reduced, or eliminated altogether by inferring more from the potentials.

References

[1] Archer, Aaron, MohammadHossein Bateni, MohammadTaghi Hajiaghayi, Howard Karloff. 2011. Improved approximation algorithms for prize-collecting Steiner tree and TSP. *SIAM Journal on Computing* **40**(2) 309–332.

[2] Arya, Sunil, Hariharan Ramesh. 1998. A 2.5-factor approximation algorithm for the k-MST problem. *Information Processing Letters* **65**(3) 117–118.

[3] Ausiello, G., M. Demange, L. Laura, V. Paschos. 2004. Algorithms for the on-line quota traveling salesman problem. *Information Processing Letters* **92**(2) 89–94.

[4] Blum, Avrim, Ramamurthy Ravi, Santosh Vempala. 1996. A constant-factor approximation algorithm for the k-MST problem. *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing (STOC)*. ACM, 442–448.

[5] Chekuri, Chandra, Nitish Korula. 2007. Approximation algorithms for orienteering with time windows. *arXiv preprint arXiv:0711.4825* .

[6] Chekuri, Chandra, Nitish Korula, Martin Pál. 2012. Improved algorithms for orienteering and related problems. *ACM Trans. Algorithms* **8**(3) 23:1–23:27. doi:10.1145/2229163.2229167. URL http://doi.acm.org/10.1145/2229163.2229167.

[7] Chekuri, Chandra, Martin Pál. 2005. A recursive greedy algorithm for walks in directed graphs. *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. IEEE, 245–253.

[8] Chen, Ke, Sariel Har-Peled. 2006. The orienteering problem in the plane revisited. *Proceedings of the Twenty-Second Annual Symposium on Computational Geometry*. ACM, 247–254.

[9] Dilkina, Bistra, Carla Gomes. 2010. Solving connected subgraph problems in wildlife conservation. *Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems* 102–116.

[10] Feigenbaum, Joan, Christos H. Papadimitriou, Scott Shenker. 2001. Sharing the cost of multicast transmissions. *Journal of Computer and System Sciences* **63**(1) 21–41.

[11] Frederickson, Greg N., Barry Wittman. 2012. Approximation algorithms for the traveling repairman and speeding deliveryman problems. *Algorithmica* **62**(3-4) 1198–1221.

[12] Freund, Daniel, Ashkan Norouzi-Fard, Alice Paul, Shane G. Henderson, David B. Shmoys. 2017. Data-driven rebalancing methods for bike-share systems. Working Paper.

[13] Garg, N. 1996. A 3-approximation for the minimum tree spanning k vertices. *Proceedings of the 37th Annual Symposium on Foundations of Computer Science*. FOCS '96, IEEE Computer Society, Washington, DC, USA, 302–309.

[14] Garg, Naveen. 2005. Saving an epsilon: a 2-approximation for the k-MST problem in graphs. *Proceedings of the Thirty-Seventh Annual ACM Symposium on Theory of Computing (STOC)*. ACM, 396–402.

[15] Goemans, Michel X., David P. Williamson. 1995. A general approximation technique for constrained forest problems. *SIAM Journal on Computing* **24**(2) 296–317.

[16] Gupta, Anupam, Ravishankar Krishnaswamy, Viswanath Nagarajan, R. Ravi. 2012. Approximation algorithms for stochastic orienteering. *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 1522–1538.

[17] Johnson, David S., Maria Minkoff, Steven Phillips. 2000. The prize collecting Steiner tree problem: theory and practice. *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. Society for Industrial and Applied Mathematics, 760–769.

[18] Kaspi, Mor, Tal Raviv, Michal Tzur. 2016. Detection of unusable bicycles in bike-sharing systems. *Omega* **65** 10–16.

[19] Levin, Asaf. 2004. A better approximation algorithm for the budget prize collecting tree problem. *Operations Research Letters* **32**(4) 316–319.

[20] Nagarajan, Viswanath, R. Ravi. 2012. Approximation algorithms for distance constrained vehicle routing problems. *Networks* **59**(2) 209–214.

[21] Reinelt, Gerhard. 1991. TSPLIB – a traveling salesman problem library. *ORSA Journal on Computing* 376–384.

[22] Vidyarthi, Shalabh, Kaushal K. Shukla. 2015. Approximation algorithms for P2P orienteering and stochastic vehicle routing problem. *arXiv preprint arXiv:1501.06515* .